ON Hom (\cdot, \cdot) AS B-ALGEBRAS

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Abstract

In this paper, we give an example to show that Hom(\cdot, \cdot) may not, in general, be a B-algebra. Moreover, we find conditions under which Hom(\cdot, \cdot) is a B-algebra. Also, we introduce the notion of an orthogonal subset and discuss some related properties.

1. Introduction

Iseki and Tanaka introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [4, 5]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [2, 3] Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [9] the authors introduced the notion of d-algebras, which is another useful generalization of BCK-algebras, and then they investigated several relations between d-algebras and BCK-algebras as well as some other interesting relations between d-algebras and oriented digraphs. Jun et al. [7] introduced a new notion, called BH-algebras, which is a generalization of BCH, BCI, BCK-algebras. They also defined the notions of ideals in BH-algebras. Recently Neggers and Kim [10] introduced the notion of B-algebra, and then Cho and Kim [1] studied some of its properties. In [6] Jun and Meng investigated some properties of Hom(X, Y) the set of all homomorphisms of a BCI-algebra X into an arbitrary BCI-algebra Y.

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algebra $Y$. In this paper, we investigate some properties of $\text{Hom}(X, Y)$ as $B$-algebras. We show that $\text{Hom}(X, Y)$ may not, in general, be a $B$-algebra for an arbitrary $B$-algebra, and we prove that if $X$ is a $B$-algebra and $Y$ is an associative $B$-algebra, then $\text{Hom}(X, Y)$ is an associative $B$-algebra. Also, we prove that if $X$ is a $B$-algebra and $Y$ is a 0-commutative $B$-algebra, then $\text{Hom}(X, Y)$ is a 0-commutative $B$-algebra. Also, we introduce the notion of orthogonal subsets and investigate some related properties.

2. Preliminaries

**Definition 2.1** [10]. A $B$-algebra is a nonempty set $X$ with a constant 0 and a binary operation $*$ satisfying the following axioms:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $(x * y) * z = x * (z * (0 * y))$,

for all $x, y, z \in X$.

**Proposition 2.2** [10]. If $(X, *, 0)$ is a $B$-algebra, then

(1) $(x * y) * (0 * y) = x$,

(2) $x * (y * z) = (x * (0 * z)) * y$,

(3) $x * y = 0$ implies $x = y$,

(4) $0 * (0 * x) = x$,

for all $x, y, z \in X$.

**Theorem 2.3** [10]. $(X, *, 0)$ is a $B$-algebra if and only if it satisfies the following axioms:

(5) $x * x = 0$,

(6) $0 * (0 * x) = x$,

(7) $(x * z) * (y * z) = x * y$,

(8) $0 * (x * y) = y * x$,

for all $x, y, z \in X$. 
Definition 2.4 [8]. A $B$-algebra $(X, \ast, 0)$ is said to be 0-commutative if
\[ x \ast (0 \ast y) = y \ast (0 \ast x), \]
for all $x, y \in X$.

Proposition 2.5 [8]. If $(X, \ast, 0)$ is a 0-commutative $B$-algebra, then

(9) $(0 \ast x) \ast (0 \ast y) = y \ast x,$
(10) $(z \ast y) \ast (z \ast x) = x \ast y,$
(11) $(x \ast y) \ast z = (x \ast z) \ast y,$
(12) $(x \ast (x \ast y)) \ast y = 0,$
(13) $(x \ast z) \ast (y \ast t) = (t \ast z) \ast (y \ast x),$

for all $x, y, z, t \in X$.

A $B$-algebra $X$ is said to be associative if $(x \ast y) \ast z = x \ast (y \ast z)$, for all $x, y, z \in X$. A nonempty subset $S$ of $X$ is called a subalgebra of $X$ if $x \ast y \in S$, for all $x, y \in S$.

Definition 2.6 [10]. A nonempty subset $N$ of a $B$-algebra $X$ is said to be normal subalgebra of $X$ if
\[ (x \ast a) \ast (y \ast b) \in N, \]
for any $x \ast y, a \ast b \in N$.

A mapping $f : x \rightarrow y$ between $B$-algebras $X$ and $Y$ is called a homomorphism if $f(x \ast y) = f(x) \ast f(y)$, for all $x, y \in X$. Define the trivial homomorphism $0$ as $0(x) = 0$, for all $x \in X$. Denote by $\text{Hom}(X, Y)$ the set of all homomorphisms of a $B$-algebra $X$ into a $B$-algebra $Y$ (see [11]).

3. $\text{Hom}(\ast, \ast)$ as $B$-algebras

Let $\text{Hom}(X, Y)$ be the set of all homomorphisms of a $B$-algebra $X$ into a $B$-algebra $Y$. In the following example, we show that $(\text{Hom}(X, Y), \ast, 0)$ may not be a $B$-algebra in general, where $\ast$ is defined as follows:
\[ (f \ast g)(x) = f(x) \ast g(x), \quad \forall f, g \in \text{Hom}(X, Y), \forall x \in X, \]
and $0$ is a trivial homomorphism from a $B$-algebra $X$ into a $B$-algebra $Y$. 
Example 3.1. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a $B$-algebra with Cayley table (Table 1) as follows:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
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<td>1</td>
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<td>1</td>
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<tr>
<td>5</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a map $f : X \to X$ by $f(x) = 0$, for all $x \in X$, and a map $g : X \to X$ by $g(x) = x$, for all $x \in X$. Then it is easily checked that $f, g \in Hom(X, Y)$, but $f \ast g \notin Hom(X, Y)$ for

$$(f \ast g)(3 \ast 1) = (f \ast g)(4) = f(4) \ast g(4) = 4$$

and

$$(f \ast g)(3) \ast (f \ast g)(1) = (f(3) \ast g(3)) \ast (f(1) \ast g(1)) = 3 \ast 2 = 5,$$

therefore,

$$(f \ast g)(3 \ast 1) \neq (f \ast g)(3) \ast (f \ast g)(1).$$

Hence, $Hom(X, Y)$ is not a $B$-algebra.

Theorem 3.2. If $X$ is a $B$-algebra and $Y$ is an associative $B$-algebra, then $Hom(X, Y)$ is an associative $B$-algebra.

Proof. Let $f, g \in Hom(X, Y)$ and $x \in X$. Then

$$(f \ast g)(x \ast y) = f(x \ast y) \ast g(x \ast y)$$

$$= (f(x) \ast f(y)) \ast (g(x) \ast g(y))$$

$$= f(x) \ast ((f(y) \ast g(x)) \ast g(y))$$

$$= (f(x) \ast (0 \ast g(y))) \ast (f(y) \ast g(x)) \quad \text{by (2)}$$
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\[
= ((f(x) \ast 0) \ast g(y)) \ast (f(y) \ast g(x)) \\
= (f(x) \ast g(y)) \ast (f(y) \ast g(x)) \text{ by (II)} \\
= (f(x) \ast (g(y) \ast f(y))) \ast g(x) \\
= f(x) \ast (g(x) \ast (0 \ast (g(y) \ast f(y)))) \text{ by (III)} \\
= f(x) \ast (g(x) \ast (f(y) \ast g(y))) \text{ by (III)} \\
= (f(x) \ast g(x)) \ast (f(y) \ast g(y)) \\
= (f \ast g)(x) \ast (f \ast g)(y).
\]

Then \( f \ast g \in \text{Hom}(X, Y) \), for all \( f, g \in \text{Hom}(X, Y) \). Since \( Y \) is a \( B \)-algebra, it is easy to prove that the axioms in Definition 2.1 are satisfied for all \( f, g, h \in \text{Hom}(X, Y) \), and so \( \text{Hom}(X, Y) \) is a \( B \)-algebra. Now let \( f, g, h \in \text{Hom}(X, Y) \) and \( x \in X \). Then

\[
((f \ast g) \ast h)(x) = (f(x) \ast g(x)) \ast h(x) = f(x) \ast (g(x) \ast h(x)) = (f \ast (g \ast h))(x),
\]

because \( Y \) is an associative \( B \)-algebra, and the proof is completed.

**Theorem 3.3.** If \( X \) is a \( B \)-algebra and \( Y \) is a 0-commutative \( B \)-algebra, then \( \text{Hom}(X, Y) \) is a 0-commutative \( B \)-algebra.

**Proof.** Let \( f, g \in \text{Hom}(X, Y) \) and \( x \in X \). Then

\[
(f \ast g)(x \ast y) = f(x \ast y) \ast g(x \ast y) \\
= (f(x) \ast f(y)) \ast (g(x) \ast g(y)) \\
= (g(y) \ast f(y)) \ast (g(x) \ast f(x)) \text{ by (13)} \\
= (0 \ast (f(y) \ast g(y))) \ast (0 \ast (f(x) \ast g(x))) \text{ by (8)} \\
= (f(x) \ast g(x)) \ast (f(y) \ast g(y)) \text{ by (9)} \\
= (f \ast g)(x) \ast (f \ast g)(y).
\]
Therefore, \( f \ast g \in \text{Hom}(X, Y) \), for all \( f, g \in \text{Hom}(X, Y) \). Since \( Y \) is a \( B \)-algebra, it is easy to prove that the axioms in Definition 2.1 are satisfied for all \( f, g, h \in \text{Hom}(X, Y) \), and so \( \text{Hom}(X, Y) \) is a \( B \)-algebra. Let \( f, g \in \text{Hom}(X, Y) \) and \( x \in X \).

Then

\[
((f \ast 0) \ast g)(x) = (f(x) \ast 0) \ast g(x) = g(x) \ast (0 \ast f(x)) = ((g \ast 0) \ast f)(x),
\]

because \( Y \) is a 0-commutative \( B \)-algebra, and the proof is completed.

**Definition 3.4.** Let \( M \) and \( \Theta \) be subsets of \( X \) and \( \text{Hom}(X, Y) \), respectively. We define orthogonal subsets \( M^\perp \) and \( \Theta^\perp \) of \( M \) and \( \Theta \), respectively, by

\[
M^\perp = \{ f \in \text{Hom}(X, Y) | f(x) = 0, \text{ for all } x \in M \}
\]

and

\[
\Theta^\perp = \{ x \in X | f(x) = 0, \text{ for all } f \in \text{Hom}(X, Y) \}.
\]

**Theorem 3.5.** Let \( X \) be a \( B \)-algebra, \( Y \) be an associative \( B \)-algebra, \( M \subseteq X \) and \( \Theta \subseteq \text{Hom}(X, Y) \). Then \( M^\perp \) and \( \Theta^\perp \) are normal subalgebras of \( \text{Hom}(X, Y) \) and \( X \), respectively.

**Proof.** Let \( f \ast g, h \ast k \in M^\perp \). Then \( (f \ast g)(x) = 0 \), for all \( x \in M \) and \( (h \ast k)(x) = 0 \), for all \( x \in M \), by Theorem 3.2, we have that \( \text{Hom}(X, Y) \) is an associative \( B \)-algebra. Thus

\[
((f \ast h) \ast (g \ast k))(x) = (((f \ast h) \ast g) \ast k)(x)
\]

\[
= (((f \ast (g \ast (0 \ast h))) \ast k)(x) \text{ by (III)}
\]

\[
= ((f \ast ((g \ast 0) \ast h)) \ast k)(x)
\]

\[
= ((f \ast (g \ast h)) \ast k)(x) \text{ by (II)}
\]

\[
= ((f \ast g) \ast (h \ast k))(x)
\]

\[
= (f \ast g)(x) \ast (h \ast k)(x) = 0,
\]

for all \( x \in M \). Thus, \( (f \ast h) \ast (g \ast k) \in M^\perp \), and so \( M^\perp \) is normal subalgebra of \( \text{Hom}(X, Y) \).
Now let \( x \ast y, a \ast b \in \Theta^\perp \), hence \( f(x \ast y) = 0 \) and \( f(a \ast b) = 0 \), for all \( f \in \text{Hom}(X, Y) \). Since \( Y \) is an associative \( B \)-algebra, in similar way we can prove that \( f((x \ast a) \ast (y \ast b)) = 0 \), for all \( f \in \text{Hom}(X, Y) \), and then \( (x \ast a) \ast (y \ast b) \in \Theta^\perp \), for all \( f \in \text{Hom}(X, Y) \). Therefore, \( \Theta^\perp \) is normal subalgebra of \( X \).

**Theorem 3.6.** Let \( X \) be a \( B \)-algebra, \( Y \) be a 0-commutative \( B \)-algebra, \( M \subseteq X \) and \( \Theta \subseteq \text{Hom}(X, Y) \). Then \( M^\perp \) and \( \Theta^\perp \) are normal subalgebras of \( \text{Hom}(X, Y) \) and \( X \), respectively.

**Proof.** Let \( f \ast g, h \ast k \in M^\perp \). Then \( (f \ast g)(x) = 0 \), for all \( x \in M \) and \( (h \ast k)(x) = 0 \), for all \( x \in M \), from Theorem 3.3 we know that \( \text{Hom}(X, Y) \) is a 0-commutative \( B \)-algebra. Hence

\[
((f \ast h) \ast (g \ast k))(x) = ((k \ast h) \ast (g \ast f))(x) \text{ by (13)}
= ((0 \ast (h \ast k)) \ast (0 \ast (f \ast g)))(x) \text{ by (8)}
= (0(x) \ast (h \ast k)(x)) \ast (0(x) \ast (f \ast g)(x)) = 0,
\]

for all \( x \in M \). Thus, \( (f \ast h) \ast (g \ast k) \in M^\perp \) and so \( M^\perp \) is normal subalgebra of \( \text{Hom}(X, Y) \).

Now, let \( x \ast y, a \ast b \in \Theta^\perp \). Then \( f(x \ast y) = 0 \) and \( f(a \ast b) = 0 \), for all \( f \in \text{Hom}(X, Y) \). Since \( Y \) is a 0-commutative \( B \)-algebra, in similar way we can prove that \( f((x \ast a) \ast (y \ast b)) = 0 \), for all \( f \in \text{Hom}(X, Y) \), and then \( (x \ast a) \ast (y \ast b) \in \Theta^\perp \), for all \( f \in \text{Hom}(X, Y) \). Therefore, \( \Theta^\perp \) is normal subalgebra of \( X \).

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**References**


