On the Study of \( ZM(G) \) the Central Measure Algebra of a Connected Lie Group

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**Abstract.** In this paper we reduce the study of connected Lie groups to the study of connected Lie groups with no compact normal semisimple subgroups. We also reduce the study of \( ZM(G) \) to the study of \( Z^G \text{M}(B(G)) \), the algebra of \( G \)-invariant bounded measures on the given characteristics subgroup \( B(G) \) of \( G \). Finally, \( B(G) \)

\[
(1) \rightarrow T^k \rightarrow B(G) \rightarrow R^m \times D \rightarrow (1)
\]

Where \( T \) is the unit circle in the complex plane and \( D \) is a discrete finitely generated abelian group.

1. Introduction and Notations

We denote the Banach space of complex, finite, regular, Borel measures on a locally compact Hausdorff space \( X \) by \( M(X) \).

If \( S \) is a locally compact topological semigroup, then \( M(S) \) is a Banach algebra under the operation \( (\mu, \nu) \rightarrow \mu * \nu \) (convolution) defined by the condition

\[
\int f \, d \mu * \nu = \int \int f(st) \, d \mu(s) \, d \nu(t) , \, f \in C_o(S)
\]

Recall that \( M(S) \) is the dual space of \( C_o(S) \), the space of continuous functions vanishing at infinity on \( S \); hence (1.1) defines a measure \( \theta * \nu \) in \( M(S) \).

It is well known that convolution is associative and distributive and satisfies \( \| \mu * \nu \| = \| \mu \| \| \nu \| \). Hence \( M(S) \) is a Banach algebra under convolution. Moreover, \( M(S) \) is commutative if and only if \( S \) is abelian. Furthermore, if \( S \) has an identity \( e \), then the point mass \( \delta_e \) at \( e \) is an identity for \( M(S) \).

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It follows from the standard facts of integration theory that (1.1) holds for all bounded Borel functions $f$ if and only if it holds for functions in $C_0(S)$. In particular, with $f = \chi_E$, the characteristics function $E$, we have

$$\mu * \nu(E) = \int \chi_E(st) \, d\mu(s) \, d\nu(t)$$

If $G$ is a locally compact group, then the measure algebra $M(G)$ has an identity $\delta_e$ and the inversion map $g \to g^{-1}$ induces an involution $\mu \to \mu^*$ on $M(G)$ defined by $\mu^*(E) = \mu(E^{-1})$.

The center $ZM(G)$ of $M(G)$ consists of all $\mu \in M(G)$ such that $\mu * \nu = \nu * \mu$ for all $\nu \in M(G)$. Then $ZM(G)$ is a commutative Banach algebra under convolution even though the underlying group $G$ need not be commutative.

2. The Operator $U^*$

Let $G$ be a locally compact group. For a compact normal subgroup $K$ of $G$, let $w_K$ be the normalized Haar measure over $K$. Hence we shall write

$$w_K(f) = \int_K f(t) \, dt, \quad f \in C_0(G)$$

**Lemma 2.1**

Let $G$ be a locally compact group, then for any compact subgroup $K$ of $G$, we have, $w_K \in ZM(G)$.

**Proof**

Since $w_K$ is the Haar measure on $K$, then $w_K$ is a Lebesgue measure on $G$, as the trivial extension of $w_K$ on $G$, and

$$\int f(tx^{-1}) \, dt = \Delta(x) \int f(t) \, dt$$

where $\Delta : G \to R$ is the continuous homomorphism modular function corresponding to $w_K$ on $G$.

Let $f = \chi_K$ be the characteristic function of $K$. We have $xKx^{-1} = K$, for each $x$ in $G$, since $K$ is a normal subgroup. But then $X_K(xtx^{-1}) = \chi_K(t)$. This implies

$$\int \chi_K(xtx^{-1}) \, dt = \int \chi_K(t) \, dt, \quad \text{for each } x \in G.$$  

Hence $\Delta(x) = 1$, for each $x$ in $G$, and

$$\int f(tx^{-1}) \, dt = \int f(t) \, dt, \quad \text{for each } f \in C_0(G).$$

**i.e.** $\delta_x * w_K * \delta_x^{-1}(f) = w_K(f)$, for each $f \in C_0(G)$ and $x$ in $G$, where $\delta_x$ is the point mass at $x$. Hence $\delta_x * w_K * \delta_x^{-1} = w_K$. So by Greenleaf, et al.[1], $w_K \in ZM(G)$. 

Corollary 2.2

For $G$, $K$ and $t$ as above, we have
\[(2.5) \int_K f(tx) \, dt = \int_K f(xt) \, dt, \quad \text{for each } x \in G \text{ and } f \in C_o(G).\]

Now let $L_x : f \to f_x = f$ (respect. $R_x : f \to f^*$) be the left (respect. right) representation of $G$, where $f_x(y) = f(xy)$ and $f^*(y) = f(yx)$. Then $L_x$ and $R_x$ can be regarded as strong continuous representations of $G$ on $L^1(G)$. So (2.5) can be rewritten as
\[(2.6) \int_K f^*(t) \, dt = \int_K f_x(t) \, dt, \quad \text{for each } f \in C_o(G) \text{ and } x \in G.

Also since the Haar measure, $w_K$ of the compact normal subgroup $K$ of $G$, is invariant under translation, we get
\[(2.7) \int_K f(t) \, dt = \int_K f_{\bar{x}}(t) \, dt, \quad \text{for each } f \in C_o(G), x \in G \text{ and } y \in \bar{x}.

where $\bar{x} = Kx = \{ tx : t \in K \}$.

Now if $f$ is a continuous function on $G$ and $x \in G$, we shall write
\[(2.8) f^*(\bar{x}) = \int_K f^*(t) \, dt.

Denote by $U^*$ the linear mapping $f \to f^*$. The following is a slight extension of a result due to Halmos[2] (see Theorem D p. 279).

Proposition 2.3

Let $G$, $K$ and $U^*$ be as above, then
\[(1) f^*$ is continuous. If $f$ is left (resp. right) uniformly continuous, so is $f^*$.
\[(2) \text{If } f \text{ is bounded, so is } f^* \text{ and } \| f^* \|_{G/K} \leq \| f \|_G.
\[(3) \text{If } f \text{ is positive definite, so is } f^*.
\[(4) \text{If } f \in C_o(G), \text{ then } f^* \in C_o(G/K) \text{ and the linear mapping}
\[U^* : C_o(G) \to C_o(G/K)

is onto.

Proof

It is easy to see that the translation map $K \times G \to G$ is continuous, since $G$ is a topological group. If $F$ is any compact set, $KF = FK$ is also compact, since $K$ is compact and normal. Hence
\[
\| f^* \|_{KF} = \sup \{ | f^*(\bar{x}) | : \bar{x} \in KF \}.
\]
\[
= \sup \{ | \int_K f(tx) \, dt | : \bar{x} \in KF \}.
\]
\[
\leq \sup \{ \int_K | f(tx) | \, dt : \bar{x} \in KF \}.
\]
\[
\leq \sup \{ | f(tx) | : t \in K \text{ and } x \in F \}
\]
\[
= \| f \|_{KF} < \infty.
\]

Applying the above relation, to $F = \{ x \}$, shows that $f^*$ is well-defined and that $f$ is bounded. Therefore, $\| f^* \|_{G/K} \leq \| f \|_G.$
It is clear that if the support of $f$ is compact, then the support of $f^*$ is contained in the compact set $KF$. For the continuity, assume first that $f$ is left uniformly continuous and let $\varepsilon > 0$. Then there is a compact neighbourhood $V$ of $e$ in $G$ such that $y^{-1}x \in V$ implies $|f(y) - f(x)| < \varepsilon$. Now for each $z \in G$ we have $(zy)^{-1}x = y^{-1}xeV$, i.e. $|f(y) - f(x)| < \varepsilon$ implies $|f(zy) - f(zx)| < \varepsilon$, for each $z \in G$. So $y^{-1}xeV$ implies

$$|f^*(y) - f^*(x)| = \left| \int_K (f(ty) - f(tx)) \, dt \right| = \int_K |f(ty) - f(tx)| \, dt \leq \int_K \varepsilon \, dt = \varepsilon$$

Suppose now $f$ is merely continuous and let $x \in G$. If $W$ is a fixed compact neighbourhood of $x$, then $V = KW$ is a compact neighbourhood of $x$ such that $tx \in V$, for each $t \in K$. By Urysohn’s lemma, there is a (uniformly) continuous function $g$ with compact support, which is equal to $f$ on $V$, hence

$$g^*(x) = \int_K g(tx) \, dt = \int_K f(tx) \, dt = f^*(x)$$

Since $g^*$ is uniformly continuous, $f^*$ is continuous at $x$.

It is obvious from the definition that if $f$ is positive definite, then so is $f^*$. It is easy now to see that whenever $f$ vanishes at infinity then so does $f^*$, i.e. $feC_0(G)$ implies $f^*eC_0(G/K)$.

Let $Q$ be a Banach algebra, $I$ be an ideal of $Q$ and $Z(Q)$ be the subalgebra of all central elements of $Q$. If $\Delta Q$ (resp. $\Delta I$) is the spectrum of $Q$ (resp. $I$) i.e. the set of all non-zero homomorphisms of $Q$ (resp. $I$) onto $C$, then we have

**Lemma 2.4**

Let $Q$ be a Banach algebra and $I$ be an ideal of $Q$. Then

$$\Delta I = \{ h \in \Delta Q : h(I) \neq 0 \}$$

**Proof**

Let $h \in \Delta I$, then there is a $j \in I$ such that $h(j) = 1$ (otherwise $h(I) = 0$). So, for every $a$ in $Q$, define $h_Q(a) = h(aj)$. Then it is easy to check that $h_Q$ is a homomorphism of $Q$, and where $h_{Q/I} = h$. So the restriction map $\Delta Q \rightarrow \Delta I$ is onto and the lemma follows easily.

Suppose now that $U : M(G/K) \rightarrow M(G)$ is the adjoint mapping of $U^* : C_0(G) \rightarrow C_0(G/K)$, i.e. for any measure $\lambda \in M(G/K)$ and any continuous function $f \in C_0(G)$, we have $U(\lambda)(f) = \lambda(f^*)$. In other words, if $\mu = U(\lambda) \in M(G)$ and $feC_0(G)$, we get

$$\int_G f(x) \, d\mu(x) = \int_{G/K} \int_K f(tx) \, dt \, d\lambda(x) \tag{2.9}$$

We conclude this section by the following theorem which we need later.

**Theorem 2.5**

Let $G$ be a locally compact group and $K$ be a compact normal subgroup of $G$. Then there exists a Banach algebra homomorphism which maps $M(G/K)$ into $M(G)$ and
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maps $\delta_K$, the identity of $M(G/K)$, to $w_K \epsilon M(G)$, the Haar measure of $K$. Moreover, we have

(1) $M(G/K) = M(G) \ast w_K$, hence $M(G/K)$ is an ideal of $M(G)$.
(2) $ZM(G/K) = ZM(G) \ast w_K$, hence $ZM(G/K)$ is an ideal of $ZM(G)$.
(3) $\Delta M(G/K) = \{ h \in \Delta M(G) : h(w_K) = 1 \}$.
(4) $\Delta ZM(G/K) = \{ h \in \Delta ZM(G) : h(w_K) = 1 \}$.

Proof

Suppose that $U : M(G/K) \rightarrow M(G)$ is the adjoint mapping of $U^* : C_o(G) \rightarrow C_o(G/K)$, see prop. 2.3 and equation (2.9) above. Since $U^*$ is a linear mapping which is onto this implies that $U$ is a linear mapping and that $U$ is one-to-one.

To show that $U$ preserves the convolution, one needs to observe that whenever $\mu = U(\lambda)$ for some $\lambda \in M(G/K)$, then $\mu(f) = \lambda(f^*)$ for all $f$ in $C_o(G)$, i.e. (2.9) is given. Now for $\lambda_1, \lambda_2 \in M(G/K)$, let $\mu_1, \mu_2 \in M(G)$ such that $\mu_1 = U(\lambda_1)$ and $\mu_2 = U(\lambda_2)$. Also let $\mu = U(\lambda_1 \ast \lambda_2)$. So we need to prove that $\mu = \mu_1 \ast \mu_2$, i.e. $U(\lambda_1 \ast \lambda_2) = U(\lambda_1) \ast U(\lambda_2)$.

For this purpose, let $f \in C_o(G)$, then

$$
\mu_1 \ast \mu_2(f) = \int_G \{ \int_G f(x) \, d\mu_1(x) \} \, d\mu_2(y) = \int_G \{ \int_G f^*(x) \, d\mu_1(x) \} \, d\mu_2(y)
$$

Let $g(y) = \int_G f^*(x) \, d\mu_1(x)$, thus, by definition of $\mu_1$, one gets

$$
g(y) = \int_{G/K} \{ \int_K f^*(tx) \, dt \} \, d\lambda_1(\overline{x}) = \int_{G/K} \{ \int_K f(txy) \, dt \} \, d\lambda_1(\overline{x})
$$

It is easy to see that $g(sy) = g(y)$, for each $s$ in $K$, since the Haar measure is translation invariant modulo $K$.

Now

$$
\mu_1 \ast \mu_2(f) = \int_G \{ \int_G f^*(x) \, d\mu_1(x) \} \, d\mu_2(y) = \int_G g(y) \, d\mu_2(y)
$$

Since the Haar measure is normalized and $g$ is $K$-invariant. Hence

$$
\mu_1 \ast \mu_2(f) = \int_{G/K} \int_{G/K} f^*(\overline{xy}) \, d\lambda_1(\overline{x}) \, d\lambda_2(\overline{y})
$$

And since $xy = \overline{xy}$, we get

$$
\mu_1 \ast \mu_2(f) = \int_{G/K} \int_{G/K} f^*(\overline{xy}) \, d\lambda_1(\overline{x}) \, d\lambda_2(\overline{y}) = (\lambda_1 \ast \lambda_2)(f^*) = U(\lambda_1 \ast \lambda_2)(f)
$$

But $f$ is an arbitrary continuous function in $C_o(G)$, so we have

$$
U(\lambda_1) \ast U(\lambda_2) = \mu_1 \ast \mu_2 = U(\lambda_1 \ast \lambda_2)
$$
Hence $U$ is a Banach algebra homomorphism.

(1) It is easy to see that for any measure $\nu$, $\nu \in M(G) \ast w_K = \{ \mu \ast w_K : \mu \in M(G) \}$ if and only if $\nu = \nu \ast w_K$ (since $w_K$ is idempotent).

Now let $\lambda \in M(G/K)$ and $\mu = U(\lambda) \in M(G)$. Then for any $\xi \in C_o(G)$, we have

$$\mu \ast w_K(\xi) = \int_G f(\tilde{x}) d\mu(x)$$

$$= \int_G f(x) d\mu(x)$$

$$= \int_{G/K} f(\tilde{x}) d\lambda(x)$$

Let $g(x) = f^*(\tilde{x})$ for each $x$ in $\tilde{x}$, then $g$ is constant on each coset $\tilde{x} = Kx$. Also $g$ is an element of $C_o(G)$, since $f$ is. Moreover $g^*(\tilde{x}) = f^*(\tilde{x}) = g(x)$, for each $x$ in $\tilde{x}$, for each $\tilde{x}$ in $G/K$. Now we can write

$$\mu \ast w_K(\xi) = \int_{G/K} f^*(\tilde{x}) d\lambda(x)$$

$$= \int_G f(x) d\mu(x)$$

$$= \mu(f)$$

But this means that $\mu \ast w_K = \mu$, i.e. $U(\lambda) \ast w_K = U(\lambda)$ for any $\lambda \in M(G/K)$. So $U(M(G/K)) = M(G) \ast w_K$ and hence $M(G/K)$ can be regarded as an ideal of $M(G)$.

(2) Let $\lambda \in ZM(G/K)$, $x \in G$ and $f$ in $C_o(G)$ be arbitrary elements.

Then

$$\delta_x \ast U(\lambda) \ast \delta x^{-1}(f) = \int_G f(xy^{-x^{-1}}) dU(\lambda)(y)$$

$$= \int_{G/K} \int_K f(txy^{-x^{-1}}) dt d\lambda(y)$$

Since the Haar measure $w_K$ is central, we get

$$\delta_x \ast U(\lambda) \ast \delta x^{-1}(f) = \int_{G/K} \int_K f(txy^{-x^{-1}}) dt d\lambda(y)$$

$$= \int_{G/K} f^*(txy^{-1}) d\lambda(y)$$

$$= (\delta \ast \lambda \ast \delta x^{-1}(f^*)$$

As $\lambda \in ZM(G/K)$, thus $\delta x^{-1} \ast \lambda \ast \delta x^{-1} = \lambda$, and $\delta_x \ast U(\lambda) \ast \delta x^{-1}(f) = \lambda(f^*) = U(\lambda)(f)$. Therefore $U(\lambda) \in ZM(G)$. Hence we have $ZM(G/K) = ZM(G) \ast w_K$.

To prove (3) and (4), let $I = M(G/K) = M(G) \ast w_K$ and $J = ZM(G/K) \cong ZM(G) \ast w_K$. Then $I$ is an ideal of $M(G)$ and $J$ is an ideal of $ZM(G)$. Applying lemma 2.5, we get $h \in I$ (resp. $h \in J$) if and only if $h(I) \neq 0$ (resp. $h(J) \neq 0$). Hence (in both cases) $h(w_K) \neq 0$. But $w_K$ is idempotent, so $h(w_K) = 1$, as required.

3. The Reduction of $G$

In this section we use Ragozin's work[3], to reduce the general case, where $G$ is a connected Lie group, to the case where a connected Lie group has no normal compact semisimple connected subgroup.
The next theorem is proved by Iwasawa\cite{4}, theorem 2, p. 515

**Theorem 3.1**

Let $G$ be a connected topological group and $K$ a compact normal subgroup of $G$. If we denote by $Z(G, K)$ the centralizer of $K$ in $G$, we have

$$G = K \cdot Z(G, K)$$

**Lemma 3.2**

If $G$ is a connected Lie group, then $G$ contains a maximal compact normal semisimple (so connected) subgroup (which may be $(1)$).

**Proof**

Let $Q$ be a semilattice of compact normal semisimple (connected) subgroups of $G$, then $Q$ has a zero as follows: let $n$ be the maximum dimension of elements of $Q$ and $K'$ be an element of $Q$ with dimension $n$. Then if $K$ is in $Q$, $KK'$ is in $Q$ and $KK'$ contains $K'$. But $\dim(KK') \leq \dim(K')$. So $KK' = K'$.

Thus $K'$ is a zero of $Q$ and is the required maximal subgroup of $G$.

**Theorem 3.3**

If $G$ is a connected Lie group, and $K$ is the maximal compact normal semisimple subgroup of $G$, then

$$G = (K \times Z(G, K)_{0})/F$$

where $Z(G, K)_{0}$ is the connected component of $Z(G, K)$, the centralizer of $K$ in $G$, and $F$ is a finite subgroup of $Z(K)$, the center of $K$.

**Proof**

We have, see theorem 3.1 above, that $G = K \cdot Z(G, K)$, since $K$ is a compact normal subgroup. Now let $p : G \rightarrow G/K$ be the projection map. Then we have $p : Z(G, K) \rightarrow G/K$ is onto. By the open mapping theorem, it follows that $p$ is an open map. Since $Z(G, K)_{0}$ is open in $Z(G, K)$, therefore $p(Z(G, K)_{0})$ is open in $G/K$. Hence it is clear now that $G = K \cdot Z(G, K)_{0}$. We also have $K \cap Z(G, K) = Z(K)$. So $K \cap Z(G, K)_{0} = F$ is a finite subgroup of $Z(K)$.

Now we can write $G$ as the direct product $K \times Z(G, K)_{0}$ modulo $F$, i.e. $G = (K \times Z(G, K)_{0})/F$.

Now let $w_{F}$ be the Haar measure on $F$. In this case we have:

**Theorem 3.4**

For $G$, $K$ and $F$ as above

1. $ZM(G) = ZM(K \times Z(G, H)_{0}) \ast w_{F}$.
2. $\Delta ZM(G) = \{ h \in \Delta ZM(K \times Z(G, K)_{0}) : h(w_{F}) = 1 \}$. 
Proof

Since $F$ is a finite central subgroup of $K$, the maximal compact normal semisimple subgroup of $G$, therefore $F$ is a compact normal subgroup of $K \times Z(G, K)$. Applying (2) and (4) of theorem 2.6, the proof follows easily.

By 3.4, if we know $\Delta ZM(K \times Z(G, K))$, we know $\Delta ZM(G)$. But results of Ragozin\cite{5} reduce the study of $\Delta ZM(K \times Z(G, K))$ to the study of $\Delta ZM(Z(G, K))$. In fact concerning the spectrum (the maximal ideal space) of $ZM(S \times H)$, the center of the measure algebra of $S \times H$ where $S$ is a compact simple Lie group and $H$ an arbitrary locally compact group, it has been proved that:

$$\Delta ZM(S \times H) = \Delta ZM(S) \times \Delta ZM(H)$$

(This together with earlier results of Ragozin\cite{3}, yield a complete description of the spectrum $\Delta ZM(K)$ for any compact connected semisimple Lie group $(K)$.

Ragozin\cite{3}, shows that if $S$ is a compact simple Lie group, then:

$$\Delta M(S) = SU \hat{Z}$$

where $Z$ is the center of $S$ and $\hat{S}$ (resp. $\hat{Z}$) is the dual space of $S$ (resp. $Z$).

So we need to study $ZM(G)$, where $G = Z(G, K)$ is a connected Lie group with no compact normal semisimple subgroups.

Standing Assumption 3.5

In the remainder of this paper, $G$ will be a connected Lie group with no compact normal semisimple subgroups.

4. The Structure of $B(G)$

In this section as a conclusion, we reduce the study of $ZM(G)$, by Greenleaf\cite{1}, to the study of $Z^G M(B(G))$, the algebra of $G$-invariant bounded measures on $B(G)$, the subgroup of $G$ of elements with relatively compact conjugacy classes. We also define the structure of $B(G)$.

Tits in\cite{6}, theorem (1) and corollary (1), shows that if $G$ is a connected locally compact group, then $B(G)$, the subgroup of $G$ of elements with relatively compact conjugacy classes, is a closed characteristic subgroup in $G$.

If we consider $G/K$, where $K$ is the maximal connected compact normal subgroup, then, according to theorem (1) of Tits\cite{6},

$$B(G/K) = B(G/K)_0 \cdot Z(G/K)$$

Since $B(G/K)_0$ is a vector group and $Z(G/K)$ is compactly generated, therefore $B(G/K)$ is a compactly generated abelian group. Now $B(G/K)$ has no compact connected
subgroup. Hence $B(G/K) = V \times D$, where $V$ is a vector group ($R^m$ for some integer $m$) and $D$ is a discrete finitely generated abelian group on which the $G$-inner automorphisms act trivially.

Hence if $G$ is a connected Lie group, then $B(G)$ satisfies the exact sequence

$$ (1) \rightarrow K \rightarrow B(G) \rightarrow R^m \times D \rightarrow (1) $$

where $K$ is the maximal compact connected normal subgroup of $G$ and $D$ is finitely generated abelian and central in $G/K$. In our case, $G$ contains no compact normal semisimple subgroup. Thus $K = T^k$ for some integer $k$, where $T$ is the unit circle in the complex plane. But then $T^k$ is central; so we will consider $ZM(G)$ for $G$ a connected Lie group with no compact connected normal semisimple subgroup. For a $G$, the subgroup $B(G)$ satisfies the central exact sequence

$$ (1) \rightarrow T^k \rightarrow B(G) \rightarrow R^m \times D \rightarrow (1) $$

i.e. $T^k$ is central, and $B(G)/T^k \cong R^m \times D$.

Greenleaf et al. [1] proved that all finite central measures on a connected Lie group $G$ are supported on the closed subgroup $B(G)$, i.e. $ZM(G) = ZGm(B(G))$. So we need to study the algebra $ZGm(B(G))$, of $I(G)$-invariant finite measures on $B(G)$.

References

حول دراسة جبر القياسات المركزیة \( ZM(G) \) لزمرة في الموصلية

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الخلاص: في هذا البحث تم تحويل زمرة في الموصلية إلى زمرة من الموصلية التي لا تحتوي على أي زمرة جزئية عمودية ملهمة بسهولة.

وقد تم كذلك تقليص دراسة جبر القياسات المركزیة \( ZM(G) \) إلى دراسة ما هو أبسط (G-invariant) على جبر القياسات المحدودة العديدة التأثير على \( Z^{G}(B(G)) \) للزمرة الجزئية المميزة \( B(G) \).

وفي النهاية أعطيت \( B(G) \) لتحقيق خواص المتتابعة الناتجة المركزية المذكورة في البحث.